# EVALUATION OF THE APPLICABILITY OF THE MODEL OF INCOMPRESSIBLE LIQUID IN THE CALCULATION OF WATER HAMMER

G. D. Rozenberg and E. G. Leonov

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The article establishes the limits of applicability of the model of incompressible liquid for calculating water hammer. The time of closing the valve is determined at which periodic oscillations do not arise.

To evaluate the effect of the compressibility of a liquid on the magnitude of water hammer, we use the linearized equations of motion of a viscous, slightly compressible liquid in pipes [1]

$$\frac{\partial p}{\partial t} = \rho c^2 \frac{\partial w}{\partial x},$$

$$\frac{\partial p}{\partial x} = \rho\left(\frac{\partial w}{\partial t} + 2aw\right), \quad 2a = \left(\frac{\lambda_0 |w|}{2d}\right)_c. \tag{1}$$

In determining $2a$, we average over the length of the pipe $l$ and time t.

With x = 0 we will consider pressure to be constant, and the speed in the section $x = l$ in closing the valve to be linearly dependent on time. Then the initial and boundary conditions for the perturbation of pressure and speed may be represented in the form: for

$$t < 0 \quad p(x,\ 0) = w(x,\ 0) = 0, \tag{2}$$

$$x = 0 \quad p = 0,$$

for

$$x = l \quad w = \begin{cases} -w_0 \dfrac{t}{T}, & 0 \leqslant t \leqslant T, \\ -w_0, & T \leqslant t < \infty. \end{cases} \tag{3}$$

If we use the method explained in [2], we find that the solution of the stated problem in Laplace transforms has the form

$$P(x,\ s) = \rho c^2 w_0 \frac{1 - \exp(-sT)}{T} \frac{\lambda \operatorname{sh} \lambda x}{s^3 \operatorname{ch} \lambda l}, \tag{4}$$

$$V(x,\ s) = w_0 \frac{\exp(-sT) - 1}{T} \frac{\operatorname{ch} \lambda x}{s^2 \operatorname{ch} \lambda l},$$

where

$$P(x,\ s) = \int_0^\infty p(x,\ t)\exp(-st)\,dt; \quad V(x,\ s) = \int_0^\infty w(x,\ t)\exp(-st)\,dt; \quad \lambda = \frac{s}{c}\sqrt{1 + \frac{2a}{s}}.$$

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In accordance with the rule of inverting Laplace transform

$$p(x,\ t) = \frac{\rho c^2 w_0}{T}\ \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} [1 - \exp(-sT)] \frac{\lambda \operatorname{sh} \lambda x}{s^3 \operatorname{ch} \lambda l} \exp(st)\, ds, \tag{5}$$

$$w(x,\ t) = \frac{w_0}{T}\ \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} [\exp(-sT) - 1] \frac{\operatorname{ch} \lambda x}{s^2 \operatorname{ch} \lambda l} \exp(st)\, ds. \tag{6}$$

Let us examine the integral

$$A_p = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} \frac{\lambda \operatorname{sh} \lambda x}{s^3 \operatorname{ch} \lambda l} \exp(st)\, ds. \tag{7}$$

The integrand in (7) has obviously a pole of second order s = 0 and simple poles $s_n$ corresponding to the roots of the equation

$$\operatorname{ch} \lambda l = \cos i\lambda l = 0. \tag{8}$$

Since the roots of Eq.(8) are equal to

$$\lambda_n = \frac{s_n}{c} \sqrt{1 + \frac{2a}{s_n}} = \frac{2n-1}{2i}\ \frac{\pi}{l},\quad n = 1,\ 2,\ 3,\ \ldots,$$

we have

$$s_n = -a \pm i\gamma_n,\quad \gamma_n = \sqrt{\left(\frac{2n-1}{2}\ \frac{\pi c}{l}\right)^2 - a^2}. \tag{9}$$

In examining relation (9), two cases are possible:

a) $\pi c/2l > a$, i.e., all $\gamma_n$ are real numbers;

b) $\pi c/2l\ (2N - 1) < a$, $\pi c/2l\ (2N + 1) > a$, then $\gamma_n$ for n = 1, 2, ..., N are imaginary, and for n = N + 1, N + 2, ... they are real numbers.

Both in case a) and in case b) all roots of (9) are in the left half plane, i.e., Re $s_n < 0$, and in (7) we may take $\gamma = 0$.

To close the integration contour in (7), we will examine the sequence of arcs with radius $R_n + (\pi c/l)n$, n = 1, 2, 3, ... with the center at the origin of coordinates, lying to the left of the imaginary axis. Since $|s_n| = \pi c/l\ (n - 1/2)$, not one of the poles lies on the arcs with radius $R_n$. It can also be shown [2] that on the mentioned arcs

$$\lim_{n\to\infty} \left| \frac{\lambda_n \operatorname{sh} \lambda_n x}{s_n^3 \operatorname{ch} \lambda_n l} \right| = 0.$$

Then, in accordance with Jordan's lemma [3] for t > 0

$$A_p = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{\lambda \operatorname{sh} \lambda x}{s^3 \operatorname{ch} \lambda l} \exp(st)\, ds = \sum_{n=0}^{\infty} \operatorname{Res} \left[ \frac{\lambda \operatorname{sh} \lambda x}{s^3 \operatorname{ch} \lambda l} \exp(st) \right]_{s=s_n}, \tag{10}$$

where the closed contour $\Gamma$ is formed by the imaginary axis and the arc $R_n$ with $n \to \infty$.

If we calculate the residues in (10), we obtain after the respective transformations:

$$A_p = \frac{x}{c^2}\left(1 + 2at + \frac{2}{3}\ \frac{a^2x^2}{c^2} - 2\frac{a^2l^2}{c^2}\right) - \frac{32 l^3 \exp(-at)}{\pi^4 c^4} \left\{ \sum_{n=1}^{N} \frac{(-1)^n}{(2n-1)^4} \left[ \frac{a}{\zeta_n} (a^2 + 3\zeta_n^2) \operatorname{sh} \zeta_n t + (3a^2 + \zeta_n^2) \operatorname{ch} \zeta_n t \right] \times \right. \tag{11}$$

$$\left. \times \sin \frac{2n-1}{2l} \pi x + \sum_{n=N+1}^{\infty} \frac{(-1)^n}{(2n-1)^4} \left[ \frac{a}{\gamma_n} (a^2 - 3\gamma_n^2) \sin \gamma_n t + (3a^2 - \gamma_n^2) \cos \gamma_n t \right] \sin \frac{2n-1}{2l} \pi x \right\},$$

where $\zeta_n = i\gamma_n$.

With the aid of considerations similar to the ones presented above, we find that

$$A_w = \frac{1}{2\pi i} \oint_\Gamma \frac{\operatorname{ch} \lambda x}{s^2 \operatorname{ch} \lambda l} \exp(st)\, ds = \sum_{n=0}^{\infty} \operatorname{Res}\left[\frac{\operatorname{ch}\lambda x}{s^2 \operatorname{ch}\lambda l} \exp(st)\right]_{s=s_n} =$$

$$= t + \frac{ax^2}{c^2} - \frac{al^2}{c^2} - \frac{16 l^2 \exp(-at)}{\pi^3 c^2}\left[\sum_{n=1}^{N} \frac{(-1)^n}{(2n-1)^3}\left(\frac{a^2+\zeta_n^2}{\zeta_n} \operatorname{sh} \zeta_n t + \right.\right. \tag{12}$$

$$\left.\left. + 2a \operatorname{ch} \zeta_n t\right) \cos \frac{2n-1}{2l} \pi x + \sum_{n=N+1}^{\infty} \frac{(-1)^n}{(2n-1)^3}\left(\frac{a^2-\gamma_n^2}{\gamma_n} \sin \gamma_n t + 2a \cos \gamma_n t\right) \cos \frac{2n-1}{2l} \pi x\right].$$

Now, in accordance with (5), (6), and the translation property in the operative calculus of [4] we obtain the solution of the stated problem in the form

$$p(x, t) = \frac{\rho c^2 w_0}{T}[A_p(x, t) - A_p(x, t-T)], \tag{13}$$

$$w(x, t) = -\frac{w_0}{T}[A_w(x, t) - A_w(x, t-T)], \tag{14}$$

where

$$A_p(x, \tau) = \begin{cases} 0 & \tau < 0, \\ A_p(x, \tau) & \tau > 0, \end{cases} \quad A_w(x, \tau) = \begin{cases} 0 & \tau < 0, \\ A_w(x, \tau) & \tau > 0. \end{cases} \tag{15}$$

For an incompressible liquid ($c = \infty$) it follows from (1), (2), and (3) that

$$w_{\text{н}}(x, t) = \begin{cases} -w_0 \dfrac{t}{T} & 0 \leqslant t \leqslant T, \\ -w_0 & T \leqslant t < \infty, \end{cases} \tag{16}$$

$$p_{\text{н}}(x, t) = \begin{cases} \dfrac{\rho w_0 x}{T} + 2a w_0 x \dfrac{t}{T} & 0 \leqslant t < T, \\ 2a w_0 x & T < t < \infty. \end{cases} \tag{17}$$

Since expressions (11) and (12) are fairly cumbersome, we neglect the viscosity in evaluating the effect of compressibility, i.e., we put $a = 0$. Then, with $x = l$,

$$A_p(l, t) = \frac{l}{c^2} - \frac{8l}{\pi^2 c^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2n-1}{2l} \pi c t, \tag{18}$$

with $x = 0$

$$A_w(0, t) = t + \frac{8l}{\pi^2 c} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{2n-1}{2l} \pi c t.$$

The sums of the trigonometric series in (18) are equal to [5]

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2n-1}{2l} \pi c t = \frac{\pi^2}{8}\left(1 - \left|\frac{ct}{l}\right|\right) \quad -2 \leqslant \frac{ct}{l} \leqslant 2, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{2n-1}{2l} \pi c t = \begin{cases} -\dfrac{\pi^2}{8} \dfrac{ct}{l} & -1 \leqslant \dfrac{ct}{l} \leqslant 1, \\ -\dfrac{\pi^2}{8}\left(2 - \dfrac{ct}{l}\right) & 1 \leqslant \dfrac{ct}{l} \leqslant 3. \end{cases}$$

Taking into account the periodicity of the expressions (19), we represent them in a more convenient form:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2n-1}{2l} \pi c t = \frac{\pi^2}{8}\left(1 - \left|\frac{ct}{l} - 4k\right|\right), \quad 4k - 2 \leqslant \frac{ct}{l} \leqslant 4k + 2,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{2n-1}{2l} \pi ct = -\frac{\pi^2}{8} \left( 1 - \left| \frac{ct}{l} - 1 - 4m \right| \right), \tag{20}$$

$$4m - 2 \leqslant \frac{ct}{l} - 1 \leqslant 4m + 2,$$

where k, m are integers.

Then for $t < T$ we have from (13)-(15), (18), and (20) that

$$p(l, t) = \frac{\rho c w_0}{T^*} |t^* - 4k|,$$

$$w(0, t) = -\frac{w_0}{T^*} (t^* - 1 + |t^* - 1 - 4m|), \tag{21}$$

for $t > T$

$$p(l, t) = \frac{\rho c w_0}{T^*} (|t^* - 4k| - |t^* - T^* - 4k_1|),$$

$$w(0, t) = -\frac{w_0}{T^*} (T^* + |t^* - 1 - 4m| - |t^* - 1 - T^* - 4m_1|), \tag{22}$$

where $T^* = cT/l$; $t^* = ct/l$. The integers $k_1$, $m_1$ are obviously determined from the conditions

$$4k_1 - 2 \leqslant t^* - T^* \leqslant 4k_1 + 2, \ 4m_1 - 2 \leqslant t^* - 1 - T^* \leqslant 4m_1 + 2. \tag{23}$$

It follows from (13), (14), and (18) that with $t > T$, the functions $p(l, t)$, $w(0, t)$ have a period equal to $4l/c$.

Now, using (21) and (22), we will examine various cases of water hammer.

1. Direct Water Hammer, i.e., $T^* \leq 2$. Analysis of formulas (20)-(23) leads to the known conclusion that

$$\max |p(l, t)| = \rho c w_0, \ \max |w(0, t)| = 2w_0. \tag{24}$$

2. Indirect Water Hammer, i.e., $T^* > 2$. For $t^* < T^*$ we have from (20) and (21) that

$$\max p(l, t) = \frac{\rho c w_0}{T^*} \max |t^* - 4k| = 2 \frac{\rho c w_0}{T^*} = \frac{\rho c w_0}{T} \frac{2l}{c},$$

which coincides with Michaud's known formula [6].

It follows from (20), (22), and (23) that for $t^* > T^*$

$$-2 \frac{\rho c w_0}{T^*} \leqslant p(l, t) \leqslant 2 \frac{\rho c w_0}{T^*}.$$

Thus, for $T^* > 2$

$$\max |p(l, t)| = 2 \frac{\rho c w_0}{T^*} = \frac{\rho c w_0}{T} \frac{2l}{c}. \tag{25}$$

For perturbations of speed, with $t^* < T^*$, we have from (20)-(23) that

$$\max |w(0, t)| \leqslant \frac{T^* + 1}{T^*} w_0,$$

with $t^* > T^*$

$$\frac{T^* - 2}{T^*} w_0 \leqslant |w(0, t)| \leqslant \frac{T^* + 2}{T^*} w_0.$$

Consequently, for $T^* > 2$

$$\max |w(0, t)| \leqslant \frac{T^* + 2}{T^*} w_0. \tag{26}$$

For an incompressible ideal liquid ($a = 0$) we have from (16) and (17) that

$$\max|w_i(0, t)| = w_0, \quad \max|p_i(l, t)| = \frac{\rho c w_0}{T^*} \tag{27}$$

independently of whether the water hammer is direct or indirect.

It follows from (24)-(27) that with direct water hammer ($T^* \leq 2$)

$$\frac{\max|p(l, t)|}{\max|p_i(l, t)|} = T^*, \quad \frac{\max|w(0, t)|}{\max|w_i(0, t)|} = 2, \tag{28}$$

and with indirect water hammer ($T^* > 2$)

$$\frac{\max|p(l, t)|}{\max|p_i(l, t)|} = 2, \quad \frac{\max|w(0, t)|}{\max|w'_i(0, t)|} \leqslant \frac{T^* + 2}{T^*}. \tag{29}$$

Thus, the model of incompressible liquid is inapplicable for evaluating the pressure perturbations attending water hammer. In fact, when $T^* < 2$, $\max|p_i(l, t)|$ may be larger by any factor than $\max|p(l,t)|$, and when $T^* > 2$, the ratio of these values is equal to two independently of the time of stopping of the flow.

Since $p(l, t)$, $p_i(l, t)$ are the perturbations above the steady-state values of the pressure $p_0(l) = p_0$, the pressures in a compressible and an incompressible liquid are respectively equal to:

$$p^*(l, t) = p_0 + p(l, t), \quad p_i^*(l, t) = p_0 + p_i(l, t).$$

Then in accordance with (25) and (29)

$$\frac{\max|p^*(l, t)|}{\max|p_i^*(l, t)|} = 1 + \frac{1}{1 + \dfrac{p_0 T}{\rho w_0 l}}. \tag{30}$$

Formula (30) makes it possible to evaluate the applicability of the model of incompressible liquid in the calculation of pressures (but not perturbations) in dependence on the permissible error and the parameters of the process.

We note the case $T^* = 4i$, $i = 1, 2, \ldots$ . With $t^* > T^* = 4i$ we have from (20) and (23) that $4k - 2 \leq t^* \leq 4k + 2$, $4k_1 - 2 \leq t^* - 4i \leq 4k_1 + 2$, $4m - 2 \leq t^* - 1 \leq 4m + 2$, $4m_1 - 2 \leq t^* - 1 - 4i \leq 4m_1 + 2$ and hence follows directly that $k_1 + i = k$, $m_1 + i = m$ and in accordance with (22) that $p(l, t) \equiv 0$, $w(0, t) \equiv -w_0$.

Consequently, with $T^* = 4i$, residual oscillations do not originate in the pipe after the valve has been closed, and the liquid is at rest.

## NOTATION

$p(x, t)$, $w(x, t)$, perturbations of pressure and mean flow velocity, respectively, above their steady-state values; $\rho$, density of the liquid; $c$, speed of sound; $\lambda_0$, coefficient of hydraulic resistance in the Darcy–Weissbach formula; $d$, pipe diameter; $x$, running length of the pipe; $t$, time; $w_0$, mean steady-state flow velocity in the section; $T$, time of valve closure; $s$, parameter of the Laplace transform; $p_i$, $w_i$, perturbations of pressure and velocity, respectively, in the flow of an incompressible liquid.

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